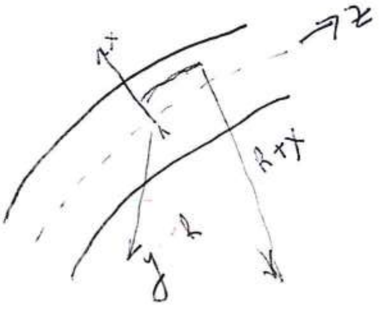


$$\frac{dz}{R} = \frac{dz'}{R+x} \qquad \frac{dz'-dz}{dz} = \frac{x}{R} = u_{zz}$$

$$\sigma_{zz} = E u_{zz} = \frac{Ex}{R}$$

Torque in crossection

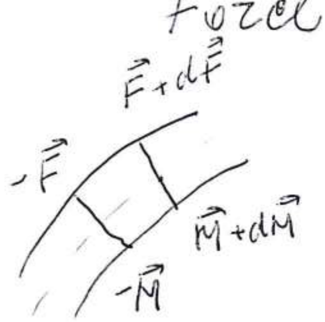
$$M_y = -\int dS\, x\, \sigma_{zz} = -\frac{E}{R}\int x^2 dS = -\frac{EI}{R}$$

Binormal $\vec{b} = [\vec{t} \times \vec{n}]$ $\qquad I = \frac{\pi a^4}{4}$

$$\vec{M} = -\frac{A}{R}\vec{b} \qquad A = EI$$

Energy $\frac{1}{2}\int u_{zz}\sigma_{zz}\, dS = \frac{EI}{2}\frac{1}{R^2} = \frac{A}{2}\frac{1}{R^2}$

Force and torque balance



$$\vec{F}+d\vec{F}-\vec{F} \Rightarrow \frac{d\vec{F}}{dl} = 0$$

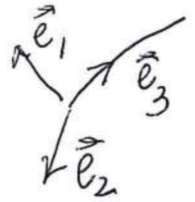
$$(\vec{r}+dl\,\vec{t}) \times (\vec{F}+d\vec{F}) - \vec{r}\times\vec{F} + \vec{M}+d\vec{M}-\vec{M} = 0 \Rightarrow$$

$$dl\,[\vec{t}\times\vec{F}] + d\vec{M} = 0$$

$$\frac{d\vec{M}}{dl} + [\vec{t}\times\vec{F}] = 0$$

Magnetic rod in an external field

$$\frac{d\vec{M}}{dl} + [\vec{t}\times\vec{F}] + [\vec{m}\times\vec{H}] = 0$$



$$\frac{d\vec{e}_3}{dl} = [\vec{\Omega} \times \vec{e}_3] \qquad \vec{\Omega} = -\frac{1}{R}\vec{e}_2$$

$$\frac{d\vec{e}_1}{dl} = [\vec{\Omega} \times \vec{e}_1]$$

$$\frac{d\vec{F}}{dl} = 0 \Rightarrow \vec{F} = \vec{F}_0$$

$$\frac{d\vec{M}}{dl} + \left[\frac{d\vec{r}}{dl} \times \vec{F}_0\right] = 0 \Rightarrow \vec{M} = \vec{M}_0 + [\vec{F}_0 \times \vec{r}]$$

$$\frac{d\vec{e}_3}{dl} = \left[\frac{\vec{M}}{A} \times \vec{e}_3\right] = \frac{1}{A}\left[(\vec{M}_0 + [\vec{F}_0 \times \vec{r}]) \times \vec{e}_3\right]$$

$$\frac{d\vec{r}}{dl} = \vec{e}_3$$

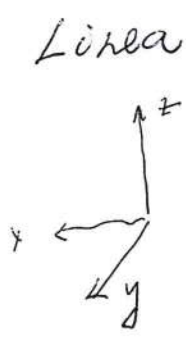
$$\vec{r} = L\tilde{\vec{r}}; \quad l = L\tilde{l}; \quad \vec{M}_0 = \frac{A}{L}\tilde{\vec{M}}_0; \quad \vec{F}_0 = \frac{A}{L^2}\tilde{\vec{F}}_0$$

$$\frac{d\vec{e}_3}{dl} = (\vec{M}_0 + [\vec{F}_0 \times \vec{r}]) \times \vec{e}_3$$

$$\frac{d\vec{r}}{dl} = \vec{e}_3$$

Linear stability analysis

$$\vec{e}_3 \simeq \left(\frac{dx}{dl}, 0, 1\right)$$



$$\vec{F}_0 = (F_{0x}, 0, F_{0z})$$

$$\vec{M}_0 = (0, M_0, 0)$$

$$\frac{dz}{dl} \simeq 1$$

$$\frac{dl_{3x}}{dl} = \frac{d^2x}{dl^2} = M_0 + F_{0z}x - zF_{0x}$$

$$x = A_1 \cos\left(\sqrt{-F_{0z}}\, l\right) + A_2 \sin\left(\sqrt{-F_{0z}}\, l\right) -$$

$$- \frac{(M_0 - l F_{0x})}{F_{0z}}$$

Fixed and clamped boundary conditions

$$x(0) = 0; \quad x(1) = 0 \qquad \frac{dx}{dl}(0) = 0; \quad \frac{dx}{dl}(1) = 0$$

$$A_1 - \frac{M_0}{F_{0z}} = 0$$

$$A_1 \cos\left(\sqrt{-F_{0z}}\right) + A_2 \sin\left(\sqrt{-F_{0z}}\right) - \frac{(M_0 - F_{0x})}{F_{0z}} = 0$$

$$\sqrt{-F_{0z}}\, A_2 + \frac{F_{0x}}{F_{0z}} = 0$$

$$-A_1 \sqrt{-F_{0z}} \sin\left(\sqrt{-F_{0z}}\right) + A_2 \sqrt{-F_{0z}} \cos\left(\sqrt{-F_{0z}}\right) + \frac{F_{0x}}{F_{0z}} = 0$$

Excluding M_0 and F_{0x}

$$\alpha = \sqrt{-F_{0z}}$$

$$A_1 (\cos\alpha - 1) + A_2 (\sin\alpha - \alpha) = 0$$

$$-A_1 \sin\alpha + A_2 (\cos\alpha - 1) = 0$$

$$\begin{vmatrix} \cos\alpha - 1 & \sin\alpha - \alpha \\ -\sin\alpha & \cos\alpha - 1 \end{vmatrix} = 0$$

$$2(1-\cos\alpha) = \alpha \sin\alpha$$

$$2\sin^2\frac{\alpha}{2} = \alpha \sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$$

1) $\sin\frac{\alpha}{2} = 0$

2) $\operatorname{tg}\frac{\alpha}{2} = \frac{\alpha}{2}$

I. $\alpha/2 = n\pi \Rightarrow$

$A_2 = 0 \qquad F_{0x} = 0$

$M_0 = -\alpha^2 A_1.$

$A_1 = ?$

$$\frac{dz}{dl} \simeq 1 - \frac{1}{2}\left(\frac{dx}{dl}\right)^2$$

$$z(1) = 1 - D$$

$$D = \frac{1}{2}\int_0^1 \left(\frac{dx}{dl}\right)^2 dl = \frac{1}{2}A_1^2\alpha^2\int_0^1 \sin^2(\alpha l)\, dl$$

$$A_1 = \frac{\sqrt{D}}{\pi n}$$

Initial configuration

$$x = A_1\left(\cos(2\pi n l) - 1\right)$$

Nonlinear case by MatLab bvp4c.

$$\frac{dx}{dl} = l_{3x}$$

$$\frac{dz}{dl} = l_{3z}$$

$$\frac{dl_{3x}}{dl} = (M_0 + F_{0z} x - z F_{0x})\, l_{3z}$$

$$\frac{dl_{3z}}{dl} = -(M_0 + F_{0z} x - z F_{0x})\, l_{3x}$$

Seven boundary conditions are necessary

$$x(0) = 0; \quad z(0) = 0; \quad x(1) = 0; \quad z(1) = 1 - \delta$$

$$l_{3x}(0) = 0; \quad l_{3z}(0) = 1; \quad l_{3z}(1) = 1$$

$$l_{3z}(1) = 1 \Rightarrow l_{3x}(1) = 0 \quad \text{since} \quad \vec{e}_3 \frac{d\vec{e}_3}{dl} = 0$$

Rod with unclamped ends.

$$\frac{d\vec{F}}{dl}=0 \Rightarrow \vec{F}=\vec{F}_0$$

$$\frac{d\vec{M}}{dl}+\left[\frac{d\vec{z}}{dl}\times\vec{F}_0\right]=0 \Rightarrow \vec{M}=\vec{M}_0+[\vec{F}_0\times\vec{z}]$$

$$\vec{M}(0)=\vec{M}(L)=0 \Rightarrow \vec{M}_0=0;\ F_{0x}=0$$

Linear theory

$$\frac{d\vec{e}_3}{dl}=\left[\frac{\vec{M}}{A}\times\vec{e}_3\right]=\frac{1}{A}[\vec{F}_0\times\vec{z}]\times\vec{e}_3$$

$$\frac{de_{3x}}{dl}=\frac{F_{0z}}{A}\times e_{3z} \qquad e_{3x}=\frac{dx}{dl}$$

$$\frac{de_{3z}}{dl}=-\frac{F_{0z}}{A}\times e_{3x} \qquad e_{3z}=\frac{dz}{dl}$$

$$e_{3z}\simeq 1$$

$$\frac{d^2x}{dl^2}-\frac{F_{0z}}{A}x=0 \qquad \tilde{F}_{0z}=\frac{F_{0z}L^2}{A}$$

$$\tilde{l}=l/L$$

$$x(0)=x(L)=0$$

$$x=A_1\sin\left(\sqrt{-\tilde{F}_{0z}}\,l\right)$$

$$\sqrt{-\tilde{F}_{0z}}=n\pi$$

Boundary conditions for non-linear case

$$x(0)=0;\ z(0)=0;\quad x(1)=0;\quad z(1)=1-\delta$$

$$+\quad e_{3x}^2(0)+e_{3z}^2(0)=1$$

$$\frac{de_{3x}}{dl}=\tilde{F}_{0z}\times e_{3z};\quad e_{3x}=\frac{dx}{dl}$$

$$\frac{de_{3z}}{dl}=-\tilde{F}_{0z}\times e_{3x} \quad e_{3z}=\frac{dz}{dl}$$

initial approximation for given $\mathcal{D}$

$$\frac{dz}{dl} \simeq 1 - \frac{1}{2}\left(\frac{dx}{dl}\right)^2$$

$$\int_0^1 \frac{dz}{dl}\, dl = 1 - \frac{1}{2}\int_0^1 A_1^2(-\tilde{F}_{0z})\cos^2\left(\sqrt{-\tilde{F}_{0z}}\, l\right) dl$$

$$1 - \mathcal{D} = 1 - \frac{1}{4}A_1^2 (n\pi)^2$$

$$A_1 = \frac{4\mathcal{D}}{\pi^2 n^2} \qquad n = 1$$

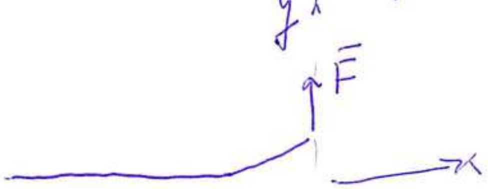
$$X = \frac{2}{\pi}\sqrt{\mathcal{D}}\, \sin(\pi l)$$

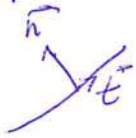
$$z = l$$

$$e_{3x} = 2\sqrt{\mathcal{D}}\cos(\pi l)$$

$$e_{3z} = 1\,; \qquad \tilde{F}_{0z} = -\pi^2.$$

Deformation of rod under applied force.





$\vec b = \vec t \times \vec n$

$\vec M = -K \frac{1}{R} \vec b$

$$\frac{d\vec M}{dl} + [\vec t \times \vec F] = 0$$

$$\vec F = K \frac{d}{dl}\left(\frac{1}{R}\right) \vec n$$

$\vec r = x \vec e_x + y(x) \vec e_y$

Small deformation

$\vec t \simeq (1, y'(x))$ $\vec n = (-y'(x), 1)$

$$\frac{1}{R} \simeq -y'' \qquad \frac{d}{dl} \simeq \frac{d}{dX}$$

$$S \frac{\partial y}{\partial t} = \frac{\partial}{\partial X} F_y = -K y_{,xxxx}$$

Laplace transform

$$\tilde y = \int_0^\infty y(t) e^{-pt} dt$$

$$\frac{\partial y}{\partial t} = p \tilde y(p) - y(0)$$

$$y(0) = 0$$

$$p \tilde y + \frac{K}{S} \tilde y_{,xxxx} = 0$$

$$\tilde y = \sum A_i e^{\alpha_i X}$$

$$\alpha_i^4 = -\left(\frac{Sp}{K}\right)$$

$Re\ \alpha_i > 0$

$$\frac{\alpha_1}{\left(\frac{5p}{K}\right)^{1/4}} = e^{\frac{i\pi}{4}}\ ;\ \frac{\alpha_4}{\left(\frac{\alpha p}{K}\right)^{1/4}} = -ie^{\frac{i\pi}{4}}$$

$$\tilde{y} = A_1 e^{e^{\frac{i\pi}{4}}\left(\frac{5p}{K}\right)^{1/4}x} + A_4 e^{-ie^{\frac{i\pi}{4}}\left(\frac{5p}{K}\right)^{1/4}x}$$

Boundary conditions

$$M_z(0) = 0 \Rightarrow y_{,xx}(0) = 0 \Rightarrow$$

$$\tilde{y}_{,xx}(0) = 0$$

$$F = -K y_{,xxx}(0) \Rightarrow$$

$$\tilde{y}_{,xxx} = -\frac{F}{pK}$$

$$A_1 i - A_2 i = 0$$

$$\left(\frac{5p}{K}\right)^{3/4} A_1 e^{i\frac{3\pi}{4}} + \left(\frac{5p}{K}\right)^{3/4} i e^{i\frac{3\pi}{4}} A_4 = -\frac{F}{pK}$$

$$A_1 = A_4 = \frac{F}{K(5/K)^{3/4}\,p^{7/4}}\frac{1}{\sqrt{2}}$$

Displacement of the end

$$y(0,t) = \frac{1}{2\pi i}\int e^{pt}dp\left(\frac{\sqrt{2}\ F}{K^{1/4}\,5^{3/4}}\frac{1}{p^{7/4}}\right)$$

(inverse Laplace transform)

By trial

$$\int_0^\infty t^\alpha e^{-pt} dt = \frac{1}{p^{\alpha+1}} \int_0^\infty e^{-t} t^\alpha dt = \frac{\Gamma(\alpha+1)}{p^{\alpha+1}}$$

$$\widetilde{t^\alpha} = \frac{\Gamma(\alpha+1)}{p^{\alpha+1}} \qquad \alpha = 3/4$$

$$y(0,t) = \frac{\sqrt{2}\, F}{K^{1/4} S^{3/4}} \frac{t^{3/4}}{\Gamma(7/4)}.$$

$$\langle y^2(0,t) \rangle = ?$$

Complex susceptibility.

$$X(\omega) = \alpha(\omega) F(\omega)$$

Notation $a = \dfrac{\sqrt{2}}{K^{1/4} S^{3/4} \Gamma(7/4)}$

$$F(t), \quad F(-\infty) = 0.$$

Linear superposition

$$y(t) = a \int_{-\infty}^{t} (t-t')^{3/4} \frac{dF}{dt'}(t')\,dt' =$$

$$= \frac{3a}{4} \int_{-\infty}^{t} (t-t')^{-1/4} F(t')\,dt'$$

$$F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$

$$y(t) = \frac{1}{2\pi} \frac{3a}{4} \int_{-\infty}^{t} (t-t')^{-1/4} \int_{-\infty}^{\infty} e^{i\omega t'} F(\omega) d\omega \, dt' =$$

$$= \frac{1}{2\pi} \frac{3a}{4} \int_{-\infty}^{\infty} F(\omega) d\omega \, e^{i\omega t} \int_{-\infty}^{t} (t-t')^{-1/4} e^{i\omega(t'-t)} dt' =$$

$$= \frac{1}{2\pi} \frac{3a}{4} \int_{-\infty}^{\infty} F(\omega) d\omega \, e^{i\omega t} \int_{0}^{\infty} \tau^{-1/4} e^{-i\omega\tau} d\tau$$

$$\alpha(\omega) = \frac{3a}{4} \int_{0}^{\infty} \tau^{-1/4} e^{-i\omega\tau} d\tau$$

$$y(t) = \frac{1}{2\pi} \int \alpha(\omega) F(\omega) e^{i\omega t} d\omega$$

$$y(\omega) = \alpha(\omega) F(\omega)$$

Complex susceptibility

$$\alpha(\omega) = \frac{3a}{4} \int_{0}^{\infty} \tau^{-1/4} e^{-i\omega\tau} d\tau$$

$$\alpha = \alpha' - i\alpha''$$

$$\langle y(t) y(0) \rangle = \frac{1}{2\pi} \int \langle |y(\omega)|^2 \rangle e^{i\omega t} d\omega$$

Fluctuation- dissipation theorem

$$\langle |y(\omega)|^2 \rangle = \frac{2k_B T \alpha''(\omega)}{\omega}$$

Mean square displacement

$$\langle (y(t)-y(0))^2\rangle = \langle y(t)^2\rangle + \langle y(0)^2\rangle - 2\langle y(t)y(0)\rangle =$$

$$= 2\,\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(1-e^{i\omega t}\right)\langle |y(\omega)|^2\rangle\, d\omega =$$

$$= 2\,\frac{1}{2\pi}\,2k_BT\,\frac{3a}{4}\int_{-\infty}^{\infty}\left(1-e^{i\omega t}\right)\int_0^{\infty}\frac{\tau^{-1/4}\sin\omega\tau}{\omega}\,d\tau\, d\omega =$$

$$= \frac{4k_BT}{2\pi}\,\frac{3a}{4}\cdot 2\int_0^{\infty}(1-\cos\omega t)\,d\omega\int_0^{\infty}\frac{\sin\omega\tau\, d\tau}{\tau^{1/4}\,\omega} =$$

$$= \frac{4k_BT}{2\pi}\,\frac{3a}{4}\cdot 2\int_0^{\infty}\frac{d\tau}{\tau^{1/4}}\int_0^{\infty}d\omega\left(\frac{\sin\omega\tau}{\omega}-\frac{\cos\omega t\sin\omega\tau}{\omega}\right)$$

$$\int_0^{\infty}\frac{\sin\omega\tau}{\omega}\,d\omega = \frac{\pi}{2}\qquad (\tau>0)$$

$$\int_0^{\infty}\frac{\cos\omega t\,\sin\omega\tau}{\omega}\,d\omega = \begin{cases}\dfrac{\pi}{2} & \tau>t\\ 0 & \tau<t\end{cases}$$

$$\langle (y(t)-y(0))^2\rangle = \frac{4k_BT}{2\pi}\,\frac{3a}{4}\cdot 2\int_0^{\infty}\frac{d\tau}{\tau^{1/4}}\left(\frac{\pi}{2}-\frac{\pi}{2}\theta(\tau-t)\right) =$$

$$= \frac{4k_BT}{2\pi}\,\frac{3a}{4}\cdot 2\cdot\frac{\pi}{2}\int_0^{t}\frac{d\tau}{\tau^{1/4}} =$$

$$= \frac{4k_BT}{2\pi}\,\frac{3a}{4}\cdot 2\,\frac{\pi}{2}\,\frac{4}{3}\,t^{3/4} =$$

$$= 2k_BT a\, t^{3/4}\ !$$

Propulsion of oscillating rod

$y = A\cos(kx + \omega t)$

wave to left with

$v = \frac{\omega}{k}$

$\vec{r} = x\,\vec{e}_x + y\,\vec{e}_y$ $\qquad \vec{t} = (1, y')$

$\vec{v} = \frac{\partial x}{\partial t}\vec{e}_x + \frac{\partial y}{\partial t}\vec{e}_y$ $\qquad \vec{n} = (-y', 1)$

$$v_n = y_{,t} - \frac{\partial x}{\partial t} y_{,x}$$

$$v_t = \frac{\partial x}{\partial t} + y_{,t}\, y_{,x}$$

Condition of inextensibility

$$\vec{t}\,\frac{d\vec{v}}{dl} = 0 \Rightarrow$$

$$\frac{dv_t}{dl} + v_n \frac{1}{R} = 0 \qquad \frac{1}{R} = -y_{,xx}$$

$$\frac{dv_t}{dx} = y_{,t}\, y_{,xx} = \frac{\partial}{\partial x}(y_{,t}\, y_{,x}) - \frac{\partial}{\partial t}\frac{1}{2}y_{,x}^2$$

$$\frac{d\bar{v}_t}{dx} = \frac{\partial}{\partial x}\overline{(y_{,t}\, y_{,x})}$$

$$\bar{v}_t = \overline{y_{,t}\, y_{,x}}$$

Mean force

$$\bar{F}_x = -\zeta_\perp \overline{v_n n_x} - \zeta_\parallel \bar{v}_t t_x =$$

$$= (\zeta_\perp - \zeta_\parallel)\,\overline{y_{,t}\, y_{,x}}$$

Total propelling force

$$F_p = \int_0^L (\zeta_\perp - \zeta_\parallel)\,\overline{y_{,t}\, y_{,x}}\,dx$$

$$F_p = \frac{(S_\perp - S_\parallel)\, A^2 k \omega L}{2}$$ in positive x axis direction!

6. Taylor sheet problem.

$$y = A \sin(kx + \omega t)$$

$$-\nabla p + \eta \Delta \vec{v} = 0 \qquad \vec{v}|\cancel{{}_{\text{sheet}} = 0}$$

$$\vec{v}(x, y) = \frac{dy}{dt} \vec{e}_y$$

$$\vec{v}_\infty = -\frac{1}{2} A^2 k \omega$$

sheet self-propels

$$v = \frac{1}{2} A^2 k \omega.$$

Balance of forces on the rod $\qquad S_\perp / S_\parallel = 2$

$$-S_\parallel L V + F_p = 0$$

$$V = \frac{A^2 k \omega}{2}$$

Some simple applications of FDT.

Brownian particle in viscous fluid.

$$\langle \Delta X^2(t)\rangle = \langle (X(t) - X(0))^2\rangle = 2\,\frac{2k_BT}{2\pi}\int_{-\infty}^{\infty}\left(1-e^{i\omega t}\right)\frac{\alpha''(\omega)}{\omega}\,d\omega$$

$$F = a\dot{X} \qquad (a = 6\pi\eta R)$$

$$\alpha(\omega) = \frac{1}{ai\omega}$$

$$\langle \Delta X^2(s)\rangle = \int_0^{\infty} e^{-st}\langle \Delta X^2(t)\rangle\,dt = \frac{2\,2k_BT}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega\,\alpha''(\omega)\,i}{s'(i\omega - s')}$$

Analytic continuation

$$s' = \varepsilon + i\omega' \qquad \varepsilon > 0$$

$$\langle \Delta X^2(i\omega')\rangle = \frac{2\,2k_BT}{2\pi}\,\frac{1}{i\omega'}\int_{-\infty}^{\infty}\frac{d\omega\,\alpha''(\omega)}{\omega-\omega'+i\varepsilon}$$

Sohotsky relation

$$\langle \Delta X^2(\omega')\rangle = \frac{2\,2k_BT}{2\pi}\,\frac{1}{i\omega'}\left(-i\pi\alpha''(\omega') + P\int\frac{\alpha''(\omega)\,d\omega}{\omega-\omega'}\right)$$

Cramers-Kroning relation

$$\alpha'(\omega') = \frac{1}{\pi}P\int\frac{\alpha''(\omega)\,d\omega}{\omega-\omega'}$$

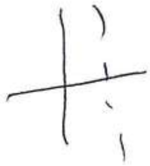
$$\Delta X^2(\omega') = \frac{2k_BT\,\alpha(\omega')}{i\omega'}$$

$$i\omega' = s'$$

$$\langle \Delta X^2(s)\rangle = \frac{2k_BT}{a}\cdot\frac{1}{s^2}$$

Inverse transform

$$\langle \Delta X^2(t)\rangle = \frac{1}{2\pi i}\int \langle \Delta X^2(s)\rangle e^{st}\, ds$$



pole $s=0$

$$\langle \Delta X^2(t)\rangle = \frac{2k_BT}{a}t$$

Fluctuations of particle in laser tweezer

$$F = a\dot{x} + KX$$

$$\alpha(\omega) = \frac{1}{K+i\omega a}$$

$$\langle \Delta X^2(s)\rangle = \frac{2k_BT}{s(K+sa)}$$

$$\langle \Delta X^2(t)\rangle = \frac{1}{2\pi i}\, 2k_BT\int \frac{e^{st}}{s(K+sa)}\, ds$$

poles $s=0$ $s=-K/a$

$$\langle \Delta X^2(t)\rangle = 2k_BT\left(\frac{1}{K} - \frac{1}{K}e^{-K/a\, t}\right)$$

$t\to\infty$

$$\langle \Delta X^2\rangle \to \frac{2k_BT}{K} \Rightarrow \langle X^2(t)\rangle = \frac{k_BT}{K}$$

$$\langle X(t)X(0)\rangle \to 0$$

$t\to 0$

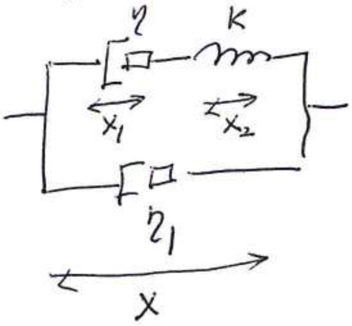
$$\langle \Delta X^2(t)\rangle = \frac{2k_BT}{a}$$ → Einstein relation

Power spectrum – important for calibration of laser tweezer

$$\langle |X(\omega)|^2\rangle = \frac{2k_BT\alpha''(\omega)}{\omega} \qquad \alpha'' = \frac{\omega a}{K^2+(\omega a)^2}$$

$$\langle |X(\omega)|^2\rangle = \frac{2k_BTa}{K^2+(\omega a)^2}$$

Random walk in viscoelastic fluid



$\eta = 0$
Voigt model

$\eta_1 = 0$
Maxwell model

What viscosity we should put in Einstein f-la?

Calculation of complex susceptibility

$$F = \eta_1 \dot{X} + K X_2$$

$$\eta \dot{X}_1 = K X_2 \; ; \quad X_1 + X_2 = X \quad \Rightarrow \qquad \tau = \eta / K$$

$$\dot{X}_2 + \frac{K}{\eta} X_2 = \dot{X}$$

$$X_2 = \int_{-\infty}^{t} e^{-\frac{(t-t')}{\tau}} \dot{X}(t')\, dt'$$

$$F = \eta_1 \dot{X} + \frac{\eta}{\tau} \int_{-\infty}^{t} e^{-\frac{(t-t')}{\tau}} \dot{X}(t')\, dt'$$

$$F = F(\omega) e^{i\omega t} \; ; \quad X(\omega) = X(\omega) e^{i\omega t}$$

$$F(\omega) = i\omega \left(\eta_1 + \frac{\eta}{1 + i\omega\tau} \right) X(\omega) = \alpha^{-1}(\omega) X(\omega)$$

$$\alpha(\omega) = \frac{1 + i\omega\tau}{i\omega \left(\eta + \eta_1 + i\omega\tau \eta_1 \right)}$$

Mean square displacement by fluctuation-dissipation theorem

$$\langle \Delta X^2(t)\rangle = \langle (X(t) - X(0))^2 \rangle = \frac{2\,2k_B T}{2\pi}\int_{-\infty}^{\infty}\left(1 - e^{i\omega t}\right)\frac{\alpha''(\omega)}{\omega}\,d\omega$$

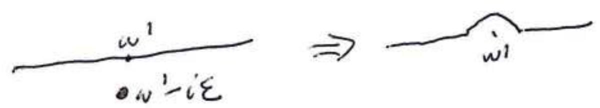
Laplace transform

$$\langle \Delta X^2(s')\rangle = \int_0^{\infty} e^{-st}\langle \Delta X^2(t)\rangle\,dt =$$

$$= \frac{2\,2k_B T}{2\pi}\int \frac{d\omega\,\alpha''(\omega)\,i}{s'\,(i\omega - s')}$$

Analytic continuation $\quad s' = \varepsilon + i\omega' \quad (\varepsilon > 0)$

$$\langle \Delta X^2(i\omega')\rangle = \frac{2\,2k_B T}{2\pi}\,\frac{1}{i\omega'}\int\frac{d\omega\,\alpha''(\omega)}{\omega - \omega' + i\varepsilon} \qquad \varepsilon \to 0$$



Sokotsky relation

$$\langle \Delta X^2(i\omega')\rangle = \frac{2\,2k_B T}{2\pi}\,\frac{1}{i\omega'}\left(-i\pi\,\alpha''(\omega') + P\int\frac{\alpha''(\omega)\,d\omega}{\omega - \omega'}\right)$$

Cramers - Kroning relation

$$\alpha'(\omega') = \frac{1}{\pi}P\int\frac{\alpha''(\omega)\,d\omega}{\omega - \omega'}$$

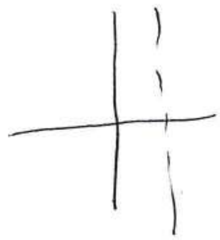
$$\langle \Delta X^2(i\omega')\rangle = 2k_B T\,\frac{\alpha(\omega')}{i\omega'}$$

$$i\omega' = s'$$

$$\langle \Delta X^2(s)\rangle = 2k_B T\,\frac{(1 + s\tau)}{s'^2\left(\eta + \eta_1 + \tau\eta_1 s'\right)}$$

Inverse transform

$$\langle \Delta X^2(t) \rangle = \frac{2k_B T}{2\pi i} \int \frac{(1+s\tau)}{s^2(\eta+\eta_1+\tau\eta_1 s)} e^{st} ds$$



Poles $\quad s=0$

$$s = -\frac{\eta+\eta_1}{\tau\eta_1}$$

Residues $\quad s=0$

$$\frac{\tau\eta}{(\eta+\eta_1)^2} + \frac{t}{\eta+\eta_1}$$

$$s = -\frac{(\eta+\eta_1)}{\tau\eta_1}$$

$$-\frac{\eta\tau}{(\eta+\eta_1)^2} e^{-\frac{(\eta+\eta_1)t}{\tau\eta_1}}$$

As a result

$$\langle \Delta X^2(t) \rangle = 2k_B T \left(\frac{\tau\eta}{(\eta+\eta_1)^2} + \frac{t}{\eta+\eta_1} - \frac{\eta\tau}{(\eta+\eta_1)^2} e^{-\frac{(\eta+\eta_1)t}{\tau\eta_1}} \right)$$

$$t \to 0$$

$$\langle \Delta X^2(t) \rangle = \frac{2k_B T}{\eta_1} t$$

$$t \to \infty$$

$$\langle \Delta X^2(t) \rangle = \frac{2k_B T}{\eta+\eta_1} t$$