Rotary motors and helices.

Helice $\vec{r} = \left(R\cos(\lambda l), R\sin(\lambda l), \frac{P}{2\pi}\lambda l \right)$

$$\left|\frac{d\vec{r}}{dl}\right| = 1 \qquad \lambda = \frac{1}{\sqrt{R^2 + (P/2\pi)^2}}$$

R - radius, $\frac{P}{2\pi}$ - pitch.

$$\vec{t} = \frac{d\vec{r}}{dl} = \left(-R\lambda \sin\lambda l, R\lambda\cos\lambda l, \frac{P\lambda}{2\pi} \right)$$

$$\frac{d\vec{t}}{dl} = \left(-R\lambda^2\cos\lambda l, -R\lambda^2\sin\lambda l, 0 \right)$$

$$K = R\lambda^2; \quad \vec{n} = (\cos\lambda l, \sin\lambda l, 0)$$

$$\frac{d\vec{n}}{dl} = K\vec{t} + \tau\vec{b} \quad (\vec{b} - \text{binormal})$$

$$\vec{b} = \left(-\frac{P\lambda}{2\pi}\sin(\lambda s), \frac{P\lambda}{2\pi}\cos\lambda s, -R\lambda \right)$$

$$\tau = \frac{P\lambda^2}{2\pi}$$

Helice rotate with angular velocity $(0, 0, \omega)$

Resistive force coefficients approximation

$$F_z = \int_0^L dl \left(-\zeta_{\parallel} t_z V_t - \zeta_{\perp} b_z V_b \right)$$

$$V_t = R^2\lambda\omega$$

$$V_b = \frac{PR\lambda}{2\pi}\omega$$

Force on the rotating helice

$$F_z = L(\varsigma_\perp - \varsigma_\parallel)\,\omega \frac{P}{2\pi} R^2 \lambda^2$$

Helice angle

$$\cos\psi = \frac{P}{2\pi}\lambda$$

Projection length l; $L = \frac{l}{\cos\psi}$

$$F_z = l(\varsigma_\perp - \varsigma_\parallel)\frac{P}{2\pi}\frac{(1-\cos^2\psi)}{\cos\psi}\,\omega$$

Torque

$$\vec{M}_z = -\int_0^L dl\,[\vec{r}\times(\varsigma_\parallel v_t \vec{t} + \varsigma_\perp v_b \vec{b})]$$

$$M_z = -\frac{l\omega}{\cos\psi}\left(\frac{P}{2\pi}\right)^2(1-\cos^2\psi)\,(\varsigma_\parallel \,tg^2\psi + \varsigma_\perp)$$

If helice moves with velocity $\vec{v} = v\vec{e}_z$ there is torque on helice

$$\cancel{\vec{M}_z = -\int_0^L dl\ \vec{r}\times(\varsigma_\parallel t_z^2 + \varsigma_\perp b_z^2)}$$

$$M_z = -\int_0^L dl\,(\varsigma_\parallel t_z[\vec{r}\times\vec{t}]_z + \varsigma_\perp b_z[\vec{r}\times\vec{b}]_z)\,v$$

$$= l\,\frac{P}{2\pi}\,\frac{(1-\cos^2\psi)}{\cos\psi}(\varsigma_\perp - \varsigma_\parallel)\,v$$

Finally friction due to motion of the helice

$$F_z = -\int_0^L dl\,(\zeta_\parallel t_z^2 + \zeta_\perp b_z^2)\,V =$$

$$= -V\frac{l}{\cos\psi}\left(\zeta_\parallel \cos^2\psi + \zeta_\perp(1-\cos^2\psi)\right)$$

F_z, M_z are force and torque on the body from the liquid

$$\begin{pmatrix} F_z \\ M_z \end{pmatrix} = \begin{pmatrix} -A & B \\ C & -D \end{pmatrix}\begin{pmatrix} v \\ \omega \end{pmatrix}$$

$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ – propulsion matrix according to Purcell, PNAS, 94, 11307 (1997)

Purcell proves $BC \geq 0 \Rightarrow B = C$

For our helice

$$B = C = \; l\,\frac{P}{2\pi}\,\frac{(1-\cos^2\psi)}{\cos\psi}(\zeta_\perp - \zeta_\parallel)\,V$$

Efficiency of propulsion according to Purcell

$$Bv - \mathcal{D}\omega + T_m = 0$$

$$-A_0 v + F = 0$$

$$-Av + B\omega - F = 0$$

$$-\mathcal{D}_0 \Omega - T_m = 0$$

$$\Omega_m = \omega - \Omega$$

Efficiency $$\mathcal{E} = \frac{A_0 v^2}{T_m \Omega_m} = \frac{A_0 \mathcal{D}_0 B^2}{\left[\mathcal{D}(A_0 + A) - B^2\right]\left[(A + A_0)(\mathcal{D} + \mathcal{D}_0) - B^2\right]}$$

$$B^2 \ll A\mathcal{D}$$

$$\frac{\mathcal{D}_0}{\mathcal{D} + \mathcal{D}_0} \sim 1$$

$$\mathcal{E} = \frac{A_0 B^2}{(A_0 + A)^2 \mathcal{D}}$$

$\mathcal{E}_{max} = \frac{B^2}{4A\mathcal{D}}$ - depends only on the shape of propeller.

Left-handed helice rotating counter-clockwise looking along z is faster.

Magnetotactic bacteria ~~are~~ have two flagella. switching ~~from~~ of directions of motion bring be connected with switch of rotating flagella.

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(Fig. 1) (Fig. 2)

Spinning flagella around bacterium

(Fig. 3)

$$-\nabla p + \eta \Delta \vec{u} + \frac{1}{2} \operatorname{rot} \vec{\tau} = 0; \quad \operatorname{div} \vec{u} = 0$$

another flow - Stokes solution for sphere moving with $\hat{\vec{u}}$

$$-\nabla \hat{p} + \eta \Delta \hat{\vec{u}} = 0; \quad \operatorname{div} \hat{\vec{u}} = 0$$

Reciprocal theorem

$$\int \hat{u}_i \sigma_{ik} n_k dS = \int u_i \hat{\sigma}_{ik} n_k dS + \int \vec{\tau} \hat{\vec{\Omega}}_0 dV$$

$$\hat{\sigma}_{ik} n_k = -\frac{3\eta}{2a} \hat{u}_i, \quad \hat{\vec{\Omega}}_0 = \frac{3a}{4} \frac{[\hat{\vec{u}} \times \vec{r}]}{r^3}$$

$$\int \sigma_{ik} n_k dS = 0$$

$$\vec{u} = \frac{1}{8\pi\eta} \int \frac{\vec{r} \times \vec{\tau}}{r^3} dV$$

Boundary layer approximation

$$u = \frac{N \tau_f}{8\pi\eta a^2} K_u$$

τ_f - torque of rotary motors

$$\vec{e}_\tau = \vec{\tau}/\tau; \quad \vec{u} = u \vec{e}_u$$

$$K_u = \frac{1}{4\pi} \int \vec{e}_u \times \vec{e}_\tau \cdot \vec{e}_r \, do$$

Equivalent approach

$$\vec{\tau} = \frac{3}{2} n \tau_f K_u \sin\vartheta \, \vec{e}_\varphi$$

Boundary layer

$$\frac{\partial \sigma_{\vartheta r}}{\partial r} - \frac{1}{2}\frac{\partial \tau_\varphi}{\partial r} = 0 \Rightarrow \sigma_{\vartheta r} = \frac{1}{2}\tau_\varphi \text{ (zero total force)}$$

$$u_\varphi(\infty) - u_\varphi(r) = \frac{1}{2\eta}\int_r^\infty \tau_\varphi \, dr'$$

Slip velocity

$$u_\vartheta| = \frac{3 N_\Sigma \tau_f K_u}{4\eta} \sin\vartheta = u_0 \sin\vartheta \text{ (squirmer model)}$$

Solution $U = \frac{2}{3} u_0 = \frac{N \tau_f}{8\pi\eta a^2} K_u$

Angular velocity

$$\Omega_i = \frac{1}{16\pi\eta}\int\left(\frac{3 r_i (\vec{\tau}\cdot\vec{r})}{r^5} - \frac{\tau_i}{r^3}\right) dV$$

Fig 2

$$\vec{e}_\Omega \cdot \vec{\Omega} = \frac{N \tau_f}{16\pi\eta a^3} K_\Omega$$

$$K_\Omega = \frac{1}{4\pi}\int \left(3(\vec{e}_r\cdot\vec{e}_\tau)(\vec{e}_\Omega \vec{e}_r) - \vec{e}_\Omega\cdot\vec{e}_\tau\right) d\omega$$

Estimates

$\tau_f = 5\,pN\cdot\mu m$; $a = 8.5\,\mu m$; $\eta = 1\,cP$; $N = 200$

$$U = \frac{N\tau_f}{8\pi\eta a^2} \simeq 550 \,\mu m/s$$

$$\Omega = \frac{N\tau_f}{16\pi\eta a^3} \simeq 32 \text{ rad/s}$$

Rotating crystals

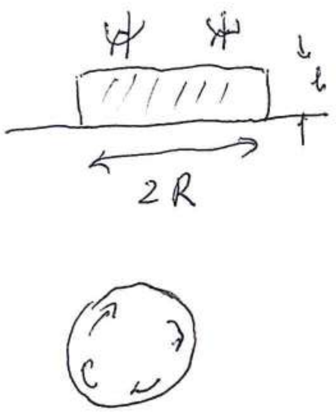

Fig. 4

$$\vec{\omega} = n\alpha\Omega_0 \vec{e}_z\, \theta(R-r)\,\theta(h-z)$$

$$\Omega_0 < 0$$

$$V_\varphi = \frac{n\alpha\Omega_0 R}{2\eta}\int_0^\infty \frac{J_1(kR)\,J_1(kr)}{k}\left(1-e^{-kz}-e^{-kh}\,\mathrm{sh}(kz)\right)dk$$

$$\Omega = V_\varphi\big|_{r=R}/R\,;\quad T = 2\pi/\Omega.$$